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DIFFERENTIAL QUADRATURE FOR STATIC AND FREE VIBRATION ANALYSES OF ANISOTROPIC PLATES

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Abstract—The differential quadrature method is used to analyse the deflection, buckling and free vibration behavior of anisotropic rectangular plates under various boundary conditions. The roots of Chebyshev polynomials are used to obtain grid-point locations and a new approach is used to apply boundary conditions. Results compare very well with existing numerical data, and less computational effort is required for the problems considered.

1. INTRODUCTION

The rectangular plate is a common structural element in many kinds of high-performance surface and air vehicles. The demand for improved structural efficiency in such applications has resulted in consideration of fiber-reinforced composite materials as the plate material (Bert, 1978). Therefore, it is important for a designer to know the fundamental coupling mechanics of such components.

Due to the complicated mathematical structure of the boundary conditions for anisotropic plates, closed form solutions exist for only a few special cases. Therefore, numerical methods must be used to obtain results. There are many computational methods available today for analysts, such as assumed-mode methods (Rayleigh–Ritz, Galerkin, etc.) and numerical methods (finite elements, finite differences, etc.). The assumed-mode methods require less computational effort as compared with finite element and finite difference methods. However, for anisotropic plates, it is not an easy task to select displacement functions satisfying all boundary conditions. If only the geometric boundary conditions are satisfied, the rate of convergence of a Rayleigh–Ritz method may be very slow for analyses of anisotropic plates (Jones, 1975; Ashton, 1970). Thus, it is necessary to seek some alternative techniques.

The differential quadrature (DQ) method, introduced by Bellman and Casti (1971) and elaborated upon further by Civan and Sliepcevich (1984), is a rather efficient numerical method for the rapid solution of linear and nonlinear partial differential equations. The method has been applied successfully to free vibrational analyses of isotropic structural components by Bert *et al.* (1988a) for the first time. Since then, a variety of problems in structural mechanics have been solved by this method (Bert *et al.*, 1988b; Striz *et al.*, 1988; Bert *et al.*, 1989; Jang *et al.*, 1989; Sherbourne and Pandey, 1991). It is found that the DQ method is quite efficient computationally.

However, there are some problems in applying the DQ method in structural mechanics. Boundary conditions cannot be satisfied exactly because two points have to be placed at each boundary point, affecting the accuracy of the solution; the DQ method cannot be used to obtain accurate high order frequencies; and results are very sensitive to grid spacings for simply-supported anisotropic plates (Sherbourne and Pandey, 1991). Thus, the choice of grid spacing becomes critical in such cases.

Recently, contributions have been made by the present authors to solve the above mentioned problems. A new approach was proposed for the application of boundary conditions such that they are satisfied exactly at the boundary grid points (Wang and Bert, 1993; Wang *et al.*, 1992; Bert *et al.*, 1993a); and harmonic differential quadrature was

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introduced to obtain accurate higher order frequencies (Striz et al., 1992). In a related paper (Bert et al., 1993b), convergence was specifically addressed. In the present paper, a set of nonuniform grid spacings is introduced. The deflection, buckling, and free vibration behavior of anisotropic plates is investigated by the DQ method and the efficiency and reliability of the method are demonstrated.

2. DIFFERENTIAL QUADRATURE

In the differential quadrature method, the derivative of a function, with respect to a space variable at a given discrete grid point, is approximated as a weighted linear sum of the function values at all of the grid points in the entire domain of that variable. Consider a function $w(x)$ of one variable in the domain (0, 1) with N discrete grid points; then one has

$$w_i' = (\mathrm{d}w/\mathrm{d}x)_i = A_{ij}w_j \quad i = 1, 2, \ldots, N, \tag{1}$$

where $w_j = w(x_j)$, A_{ij} = weighting coefficients of the first derivative, and repeated index j means summation from 1 to N.

In the original DQ method, A_{ij} are determined by requiring that eqn (1) be exact for all polynomials $x^k (k = 0, 1, 2, \ldots, N-1)$. Weighting coefficients of higher order derivatives can be computed by matrix multiplications once A_{ij} are determined. For example,

$$B_{ij} = A_{ik}A_{kj}, \tag{2}$$

$$C_{ij} = A_{ik}B_{kj}, \tag{3}$$

$$D_{ij} = A_{ik}C_{kj} = B_{ik}B_{kj}, \tag{4}$$

where B_{ij}, C_{ij} and D_{ij} are weighting coefficients of the second, third and fourth derivatives, respectively.

Normally, uniform grid spacing is used. However, as was pointed out by Sherbourne and Pandey (1991), solutions are sensitive to grid spacing for simply-supported anisotropic plates and nonuniform grid spacing yields better results. It is recognized here (Oden, 1992) that there is a similarity between the DQ and mixed collocation methods (see the Appendix for details). Since the residual in the collocation method is minimized for ordinary differential equations if the collocation points are given by the roots of the Chebyshev polynomials (Wright, 1964), the following nonuniform grid spacing is proposed,

$$x_k = \{1, [1+\cos(2i-1)(\pi/2N)]/2, 0\} \quad (i = 2, 3, \ldots, N-1; \quad k = N, N-1, \ldots, 2, 1). \tag{5}$$

For beams fixed at both ends or plates with both sides in one direction clamped, two grid points separated by a small distance δ are placed at each boundary point (Bert et al., 1988); thus, x_k takes the following form

$$x_k = \{1, 1-\delta, [1+\cos(2i-3)[\pi/2(N-2)]]/2, \delta, 0\}$$
$$(i = 3, 4, \ldots, N-2; \quad k = N, N-1, \ldots, 2, 1). \tag{6}$$

There are many other ways to determine the weighting coefficients A_{ij}. One alternative approach is provided in the Appendix to determine A_{ij} in the DQ method. Recently, Striz et al. (1992) used harmonic functions instead of polynomials in the differential quadrature method, calling it the HDQ method to distinguish it from the original DQ method. In the HDQ method, the A_{ij} are determined by requiring that eqn (1) be exact when $w(x)$ takes the following forms:

$$w(x) = \left\{1, \sin \pi x, \cos \pi x, \sin 2\pi x, \cos 2\pi x, \ldots, \sin \frac{N-1}{2}\pi x, \cos \frac{N-1}{2}\pi x\right\}, \tag{7}$$

where N is an odd number. Experience shows that the larger the N in the HDQ method, the better the solution.

Boundary conditions, other than those in eqn (6), are applied by using the new approach proposed by the present authors (Wang and Bert, 1992; Bert *et al.*, 1993a). The essence of the method is that boundary conditions are applied during formulation of the weighting coefficients. Thus, all boundary conditions are satisfied at grid points on the boundary so that the δ effect (inaccuracy caused by placing two grid points separated by a small distance δ) is eliminated.

3. APPLICATIONS

Mid-plane symmetric laminates are more popular for practical use as compared with unsymmetric laminates; thus, the applications considered here are limited to such cases. Furthermore, the simplest layup exhibiting the most anisotropy is off-axis parallel ply. For such laminates, the governing differential equation is

$$\bar{D}_{11}w_{,xxxx}+4\bar{D}_{16}w_{,xxxy}+2(\bar{D}_{12}+2\bar{D}_{66})w_{,xxyy}+4\bar{D}_{26}w_{,xyyy}+\bar{D}_{22}w_{,yyyy}$$
$$= q_0+\rho h\omega^2 w-N_x w_{,xx}-N_y w_{,yy}, \tag{8}$$

where $\bar{D}_{ij}$ are the plate stiffnesses, h is the total plate thickness, N_x and N_y are the applied compressive loads in the respective x and y directions, q_0 is the pressure, $w = w(x, y)$ is the modal deflection, ρ is the density, ω is the radian natural frequency, $(\ )_{,xyyy}$ denotes $\partial^4(\ )/\partial x\,\partial y^3$, etc., and x and y are the midplane Cartesian coordinates. For deflection analysis, only q_0 is used; for free vibration analysis, only $\rho h\omega^2 w$ is used; for buckling analysis, only $-N_x w_{,xx}$ is used for uniaxial compression and $-N_x w_{,xx}-N_y w_{,yy}$ is used for biaxial compression.

The boundary conditions applied in the analysis are

(1) Simply-supported edges

$$x = 0, a: w = 0, \quad M_x = -\bar{D}_{11}w_{,xx}-\bar{D}_{12}w_{,yy}-2\bar{D}_{16}w_{,xy} = 0, \tag{9}$$

$$y = 0, b: w = 0, \quad M_y = -\bar{D}_{12}w_{,xx}-\bar{D}_{22}w_{,yy}-2\bar{D}_{26}w_{,xy} = 0. \tag{10}$$

(2) Clamped edges

$$x = 0, a: \quad w = 0, \quad w_{,x} = 0, \tag{11}$$

$$y = 0, b: \quad w = 0, \quad w_{,y} = 0. \tag{12}$$

For cases considered in this paper, q_0 = constant and $N_x = N_y$. Applying differential quadrature to eqn (8), one has

$$\bar{D}_{11}\bar{D}^x_{ik}w_{kj}+4\bar{D}_{16}\beta\bar{C}^x_{ik}\bar{A}^y_{jm}w_{km}+2\beta^2(\bar{D}_{12}+2\bar{D}_{66})\bar{B}^x_{ik}\bar{B}^y_{jm}w_{km}$$
$$+4\bar{D}_{26}\beta^3\bar{A}^x_{ik}\bar{C}^y_{jm}w_{km}+\bar{D}_{22}\beta^4\bar{D}^y_{jk}w_{ik} = q_{ij}a^4+\bar{\omega}^2 w_{ij}-\bar{N}a^2(\bar{B}^x_{ik}w_{kj}+\beta^2\bar{B}^y_{jk}w_{ik}),$$
$$i, j = 2, 3, \ldots, (N-1) \text{ for SS}-\text{SS}-\text{SS}-\text{SS plates},$$
$$i, j = 3, 4, \ldots, (N-2) \text{ for C}-\text{C}-\text{C}-\text{C plates}, \tag{13}$$

where $q_{ij} = q_0$, $\beta = a/b$ (a and b are plate lengths in the x and y directions), $\bar{\omega}^2 = \rho h a^4\omega^2$, and $\bar{N} = N_x = N_y$, respectively. Note that appropriate boundary conditions have been applied during the formulation of the weighting coefficients.

4. RESULTS AND COMPARISONS

For simplicity, the notations used here are that DQMU denotes uniform grid spacing in the DQ method, while DQM and HDQ denote nonuniform grid spacing [see eqns (5) and (6)] in the DQ and HDQ methods, respectively. Uniform grid spacing could also be used in the HDQ method; however, experience shows that accuracy is greatly improved by using the proposed nonuniform grid spacing, especially for higher order frequencies or for cases in which the results are sensitive to grid spacing. Center deflections are analyzed by both the DQMU and DQM for anisotropic square plates, simply supported at all four edges and subjected to uniformly distributed loading. Various bending stiffness ratios and their corresponding equivalent skew angles, given by Ashton (1969), are cited in Table 1; see also Whitney (1987), p. 140. The results are given in Table 2. It can be seen that both DQMU and DQM yield more accurate solutions than the Rayleigh–Ritz solution using 49 double sine terms by Ashton (1969) for the cases considered. By using nonuniform grid spacing, the accuracy is again greatly improved. Since the size of the resulting matrix to be inverted by the DQ method is less than ($N = 7$) or equal to ($N = 9$) that of the 49-term Rayleigh–Ritz solution, the DQ method seems computationally more efficient in reaching the same accuracy.

The fundamental frequencies are obtained by the DQ and HDQ methods for the same plates as used in the deflection analysis. No known exact solutions except for $\theta = 90°$ are available. The nondimensionalized values are given in Table 3. It is seen that results by both DQ and HDQ methods are close to each other. When $\theta = 90°$, the exact solution is achieved. Experience shows that the DQ method in general yields slightly more accurate fundamental frequency values than does the HDQ method for the same grid points. However, there is no limitation on N for the HDQ method, and the larger the N, the better the solutions.

To illustrate this, a free vibration analysis is performed on a simply-supported square graphite/epoxy plate. The principal material direction is at 45° to the plate sides, and $Q_{11}/Q_{22} = 25$, $Q_{12}/Q_{22} = 0.25$ and $Q_{66}/Q_{22} = 0.5$. Nondimensionalized natural frequencies

Table 1. Plate stiffness ratios and equivalent skew angle†

Stiffness Ratio	Equivalent skew angle θ: 90°	80°	63°	60°	54°
$\bar{D}_{22}/\bar{D}_{11}$	1.000	1.000	1.000	1.000	1.000
$(\bar{D}_{12}+2\bar{D}_{66})/\bar{D}_{11}$	1.000	1.061	1.412	1.500	1.690
$\bar{D}_{16}/\bar{D}_{11}(\bar{D}_{26} = \bar{D}_{16})$	0.000	−0.174	−0.454	−0.500	−0.587

† Ashton (1969).

Table 2. Nondimensionalized center deflection $\bar{w}$ in anisotropic square plates under uniformly distributed loading (SS-SS-SS-SS; $\bar{w} = w\bar{D}_{11}/q_0a^4$)

Equivalent skew angle θ		90°	80°	63°	60°	54°
DQMU	$N = 7$	0.00406 (0.00%)	0.00411 (0.00%)	0.00436 (−1.80%)	0.00442 (−2.21%)	0.00456 (−4.20%)
	$N = 9$	0.00406 (0.00%)	0.00411 (0.00%)	0.00440 (−0.90%)	0.00447 (−1.11%)	0.00457 (−3.99%)
DQM	$N = 7$	0.00406 (0.00%)	0.00411 (0.00%)	0.00442 (−0.45%)	0.00451 (−0.22%)	0.00473 (−0.63%)
	$N = 9$	0.00406 (0.00%)	0.00411 (0.00%)	0.00445 (0.22%)	0.00455 (−0.66%)	0.00478 (0.42%)
Specially orthotropic solution†		0.00406 (0.00%)	0.00394 (−4.14%)	0.00336 (−24.32%)	0.00324 (−28.32%)	0.00301 (−36.76%)
Rayleigh-Ritz solution†		0.00406 (0.00%)	0.00408 (−0.73%)	0.00422 (−4.95%)	0.00425 (−5.97%)	0.00430 (−9.66%)
Exact†		0.00406	0.00411	0.00444	0.00452	0.00476

† Ashton (1969).

Table 3. Nondimensionalized fundamental frequency $\bar{\omega}$ of flexural vibration of anisotropic square plates (SS-SS-SS-SS, $\bar{\omega} = (\omega a^2/\pi^2)(\rho h/\bar{D}_{11})^{1/2}$)

Equivalent skew angle θ		90°	80°	63°	60°	54°
DQM ($N = 9$)		2.000	1.988	1.913	1.892	1.845
HDQ	$N = 9$	2.000	1.987	1.905	1.883	1.830
	$N = 11$	2.000	1.988	1.909	1.888	1.837

Table 4. Nondimensionalized frequencies $\bar{\omega}$ of flexural vibration of a simply supported square graphite/epoxy plate ($\bar{\omega} = (\omega a^2/\pi^2)(\rho h/\bar{D}_{11})^{1/2}$)

Mode	1	2	3	4	5	6	7	8
DQM ($N = 9$)	50.377	91.680	140.08	150.61	193.74	221.04	267.80	313.74
HDQ ($N = 9$)	49.081	91.285	141.37	148.86	203.87	225.02	271.68	301.99
HDQ ($N = 11$)	49.809	91.509	141.98	149.56	202.72	224.94	270.84	301.90

up to mode 8 are summarized in Table 4 for both DQ and HDQ methods. It can be seen that results are close to each other up to mode 4. However, the differences become larger between these two methods for higher modes. In those cases, results are more reliable by the HDQ method than the DQ method since the HDQ method has been shown to give better results for higher order frequencies (Striz *et al.*, 1992).

To show computational efficiency, natural frequencies are analysed by the DQ method for clamped square ($a = b$) and rectangular ($a = 2b$) plates composed of an orthotropic material with the principal material axis at α degrees from the x axis. The material properties are $E_L/E_T = 10$, $G_{LT}/E_T = 0.25$, and $\nu_{LT} = 0.3$. Results obtained by the DQ method with $N = 11$ (7 inner grid points in the x or y directions) are given in Table 5 with various α values for $a = b$ and in Table 6 for $a = 2b$. The Rayleigh–Ritz results obtained by Whitney (1987) with 49 terms are also shown in Tables 5 and 6. It can be seen that results by the DQ method compare well with Whitney's data up to mode 4. It should be pointed out that for both methods, the size of the matrix involved is the same (49 × 49). However, the DQ method provides a very compact procedure to be executed on a computer. On the other hand, the 49-term Rayleigh–Ritz solution involves 490 double integrations and 1176 derivatives.

Table 5. Nondimensionalized frequencies $\bar{\omega}$ of flexural vibration of clamped square anisotropic plates ($\bar{\omega} = \omega b^2\sqrt{\rho h/D}$, $D = E_L h^3/[12(1-\nu_{LT}^2 E_T/E_L)]$)

Orientation	Mode 1		Mode 2		Mode 3		Mode 4	
α	DQM†	Ritz‡	DQM	Ritz	DQM	Ritz	DQM	Ritz
0°	23.97	23.97	31.15	31.15	46.38	46.41	62.78	62.77
15°	23.09	23.10	31.51	31.52	47.62	47.65	59.45	59.46
30°	21.33	21.35	33.14	33.18	50.63	50.72	51.79	51.87
45°	20.49	20.51	34.96	35.01	46.85	47.07	52.04	52.21

† $N = 11$ and $\delta = 0.00001$.
‡ 49-term solution (Whitney, 1987).

Table 6. Nondimensionalized frequencies $\bar{\omega}$ of flexural vibration of clamped rectangular anisotropic plates ($\bar{\omega} = \omega b^2\sqrt{\rho h/D}$, $D = E_L h^3/[12(1-\nu_{LT}^2 E_T/E_L)]$, $a/b = 2$)

Orientation	Mode 1		Mode 2		Mode 3		Mode 4	
α	DQM†	Ritz‡	DQM	Ritz	DQM	Ritz	DQM	Ritz
0°	9.34	9.34	17.61	17.61	20.83	20.83	26.49	26.49
15°	9.67	9.68	17.18	17.19	22.01	22.02	26.42	26.44
26.6°	10.56	10.57	16.89	16.93	25.06	25.11	25.38	25.46
45°	13.85	13.88	17.64	17.73	23.63	23.85	31.80	31.73
60°	17.86	17.87	19.79	19.86	23.57	23.75	29.72	29.75
75°	21.26	21.27	22.30	22.32	24.65	24.69	29.03	28.94
90°	22.57	22.57	23.38	23.38	25.30	25.30	29.06	28.87

† $N = 11$ and $\delta = 0.00001$.
‡ 49-term solution (Whitney, 1987).

Table 7. Buckling coefficients, $K(=\bar{N}b^2/Q_{22}h^3)$, of an SS-SS-SS-SS square graphite/epoxy anisotropic plate under balanced biaxial compression ($N_x = N_y = \bar{N}$)

Researcher	K	Numerical method
Whitney†	8.418 ($m = n = 7$)‡ 8.556 ($m = n = 13$)	Fourier analysis
Ashton†	11.565 ($m = n = 5$) 11.060 ($m = n = 7$)	Rayleigh–Ritz method
Present authors	8.740 ($N = 7$)§ 8.574 ($N = 9$)	DQ method

† Data cited by Jones (1975).
‡ $m(n)$—number of terms in series in the $x(y)$ direction.
§ N—number of grid points in each direction.

A comprehensive study on the buckling of composite plates by the DQ method was performed by Sherbourne and Pandey (1991). A different nonuniform grid spacing, determined by trial and error, was used in that paper. To show the efficiency of the new nonuniform grid spacing and the new approach in applying the boundary conditions, two examples of buckling analysis are presented in the following.

As a first example, a rectangular ($a = 2b$) boron/epoxy plate clamped on all edges under a uniform uniaxial compressive load (N_x) is considered. The bending stiffness ratios are $\bar{D}_{22}/\bar{D}_{11} = 1$, $(\bar{D}_{12}+2\bar{D}_{66})/\bar{D}_{11} = 2.38$, $\bar{D}_{16}/\bar{D}_{11} = \bar{D}_{26}/\bar{D}_{11} = 0.69$. Let the buckling coefficient K be $N_{cr}b^2/\bar{D}_{11}$. The result obtained by the DQ method with $N = 11$ (7×7 inner grid points) is 62.864, which compares very well with Whitney's numerical result, 62.72, obtained by the Rayleigh–Ritz method using 49 terms in a series. If $N = 13$ (9×9 inner grid points) is used, the result by the DQ method is 62.63, which should be close to the exact solution, because the Rayleigh–Ritz method provides an upper bound solution in this case. As a second example, consider a simply supported, square graphite/epoxy plate under uniform balanced biaxial compression, $N_x = N_y = \bar{N} =$ constant. The material properties are $Q_{11}/Q_{22} = 25$, $Q_{12}/Q_{22} = 0.25$, and $Q_{66}/Q_{22} = 0.5$ with principal material direction at 45° to the plate side (the x axis). Results obtained by the DQ method are given in Table 7. Numerical data by Whitney and by Ashton, cited by Jones (1975), are also listed in the same table. As was pointed out by Jones, Ashton's Rayleigh–Ritz solution converged to an incorrect value. The DQ method, with the proposed nonuniform grid spacing and the new approach in applying boundary conditions, yields reliable values of buckling load close to Whitney's data (169 terms). It should be pointed out that the DQ method with $N = 9$ needs much less computational effort than Whitney's 169-term Fourier analysis and is an alternative approach to solving the problem accurately.

5. CONCLUSIONS

The differential quadrature method, with the proposed nonuniform grid spacing, is a means to accurately determine the static and free vibrational behavior of anisotropic rectangular plates having various boundary conditions. The method is more attractive than the classical Ritz method due to its compactness and computational efficiency and shows good promise for further development as a practical computational technique.

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APPENDIX. SIMILARITY OF THE DQ METHOD WITH THE MIXED COLLOCATION METHOD

Let

$$w = \phi_j(x)w_j, \tag{A1}$$

where the repeated index j means summation from 1 to N;

$$w_j = w(x_j) \quad j = 1, 2, \ldots, N \tag{A2}$$

$\phi_j(x)$ are Lagrangian polynomials, and

$$\phi_j(x_i) = \delta_{ij} \quad i, j = 1, 2, \ldots, N. \tag{A3}$$

Thus,

$$w_i' = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x_i} = \phi_j'(x_i)w_j \quad i = 1, 2, \ldots, N. \tag{A4}$$

In the DQ method, one has

$$w_i' = A_{ij}w_j \quad i = 1, 2, \ldots, N, \tag{A5}$$

where the A_{ij} are the weighting coefficients of the first derivative and are determined by requiring that eqn (A5) be exact for all polynomials of degree less than or equal to $(N-1)$.

Comparing eqn (A4) with eqn (A5), one obtains

$$A_{ij} = \phi_j'(x_i) \quad i, j = 1, 2, \ldots, N. \tag{A6}$$

In other words, eqn (A6) can be used to find the weighting coefficients A_{ij}.

Weighting coefficients of the second, third and fourth order derivatives, B_{ij}, C_{ij} and D_{ij}, can be obtained in a similar way, namely,

$$B_{ij} = \phi_j''(x_i), \tag{A7}$$

$$C_{ij} = \phi_j'''(x_i), \tag{A8}$$

$$D_{ij} = \phi_j^{\mathrm{IV}}(x_i), \tag{A9}$$

$$i, j = 1, 2, \ldots, N.$$

In general, eqn (A1) does not satisfy the slope, bending moment, and shear force boundary conditions; thus, the original DQ method is equivalent to a mixed collocation method if the grid points and the way of applying the boundary conditions are exactly the same. However, boundary conditions are always applied at boundary points in the mixed collocation method. To the authors' knowledge, no one has used a trial solution similar to eqn (A1) in the mixed collocation method.